First, note that we know the whole system, so we can simply enumerate all configurations and maximize the entropy of the prob. distibution.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>↑↑</th>
<th>↑↓</th>
<th>↓↑</th>
<th>↓↓</th>
<th>↓↑</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplicity</td>
<td>2</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Energy $h_i$</td>
<td>4J</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-4J</td>
</tr>
<tr>
<td>Numbers (i)</td>
<td>1,2</td>
<td>3-10</td>
<td>11-14</td>
<td>15,16</td>
<td></td>
</tr>
</tbody>
</table>

Now maximize $S[P] = \sum_i p_i \ln \frac{1}{p_i}$

subject to

\[
\begin{align*}
    n &= \sum_i h_i p_i, \\
    1 &= \sum_i p_i.
\end{align*}
\]

Equivalent to maximizing the unconstrained Lagrangian

\[
    L = \sum_i p_i \ln \frac{1}{p_i} + \beta(\sum_i h_i p_i) + \lambda(1 - \sum_i p_i).
\]

\[
    0 = \frac{\partial L}{\partial p_i} = -1 - p_i - \beta h_i - \lambda
\]

\[
    \Rightarrow p_i = e^{-\mu - \beta h_i}, \quad \text{where} \quad \mu = 1 + \lambda.
\]
5.1 cont.

Now note that \( p_i \) depends only on \( \beta, M, h_i \). All configurations with equal energy must also have equal probability: \( p_2 = p_3 = \ldots = p_{14} \), \( p_{15} = p_{16} \).

Also note that \( \frac{p_1}{p_{15}} = e^{-8Jp} \), \( \frac{p_3}{p_{15}} = e^{-4Jp} \).

\[
0 = \frac{\partial L}{\partial \mu} = 1 - \sum_i p_i = 1 - p_{15} \left( 2e^{-8Jp} + 12e^{-4Jp} + 2 \right)
\]

\[
\Rightarrow \left\{ \begin{array}{l}
p_{15} = \frac{1}{2(2e^{-8Jp} + 6e^{-4Jp} + 1)} \, , \\
p_1 = p_2 = p_{15} e^{-8Jp} \, , \\
p_3 = \ldots = p_{14} = p_{15} e^{-4Jp} \, .
\end{array} \right.
\]

(And \( \beta = \frac{1}{k_B T} \) if you want to express the solution in temperature.)

**Note.** As \( T \to 0 \), \( \beta \to +\infty \) and

\[
p_{15} = p_{16} \to \frac{1}{2} \, .
\]

As \( T \to \infty \), \( \beta \to 0 \) and

\[
p_i \to \frac{1}{16} \, .
\]
these two are correlated.

we need to maximize $A S_4$.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>↑</th>
<th>↑</th>
<th>↓→</th>
<th>↓→</th>
<th>↓→</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplicity</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Energy</td>
<td>$-J$</td>
<td>$+J$</td>
<td>$+J$</td>
<td>$-J$</td>
<td></td>
</tr>
<tr>
<td>Probability</td>
<td>$p_0$</td>
<td></td>
<td>$p_1$</td>
<td></td>
<td>$p_2$</td>
</tr>
</tbody>
</table>

This is essentially three shifted/overlaid one-dimensional Ising models as in the lecture notes. Stationary solution must be the same, but scaled by $\frac{1}{4}$:

$$P_0 = P_2 = \frac{1}{8(1+e^{-2J})}, \quad P_1 = \frac{1}{8(1+e^{-2J})}$$
Maximize \( s \leftrightarrow \maximize \lim_{m \to \infty} \Delta S_m \leftrightarrow \maximize \Delta S_2 \), since we only need correlations in blocks of length 2.

So maximize \( S_2 - S_1 \).

\[
\begin{array}{c|ccc}
\text{Conf.} & \uparrow & \rightarrow & \downarrow \\
\text{Multiplicity } n_i & 4 & 8 & 4 \\
\text{Energy } h_i & -2 & 0 & 2 \\
\text{Probability } & p_1 & p_2 & p_3 \\
\end{array}
\]

\( \Sigma = 16 \)

Note that if all symmetries are preserved all "rotations" must be equally likely, so then we only have three distinct probabilities.

And due to symmetry all 1-blocks \( (\uparrow, \rightarrow, \downarrow, \leftarrow) \) are equally likely, so \( S_1 = \log 4 \) unavoidably.

Thus, \( \max \Delta S_2 \)

subject to

\[
\begin{align*}
\{ & u = \sum_i p_i n_i h_i; \\
& l = \sum_i p_i n_i; \\
\} & \end{align*}
\]

Is equivalent to unconstrained maximization of

\[
L = \sum_i p_i n_i \ln \frac{l}{p_i} + \varphi (u - \sum_i p_i n_i h_i) + (\mu - 1) (1 - \sum_i p_i n_i).
\]
\[ p_i = e^{-m \cdot \phi_i} \quad \text{(since } 0 = \frac{\partial L}{\partial p_i}) \]

\[ \frac{p_1}{p_3} = e^{2\delta p}, \quad \frac{p_2}{p_3} = e^{\delta p} \]

\[ \Rightarrow 0 = \frac{\partial L}{\partial \mu} = 1 - \sum p_i \cdot n_i = 1 - 4 \left( e^{2\delta p} + 2e^{\delta p} + 1 \right) p_3 \]

\[ \Rightarrow p_3 = \frac{1}{4 \left( e^{2\delta p} + 2e^{\delta p} + 1 \right)} \]

As \( T \to 0 \), \( p_3, p_2 \to 0 \) and \( p_1 \to \frac{1}{4} \).

As \( T \to \infty \), \( p_1, p_2, p_3 \to \frac{1}{16} \).

When \( T \to 0 \), \( \bar{\psi}_2 \to \log 4 \Rightarrow s \to 0 \).

Physical interpretation: we tend towards solutions like

\[ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \cdots \]

\[ \to \to \to \to \to \to \cdots \]
configuration
<table>
<thead>
<tr>
<th></th>
<th>AA</th>
<th>BB</th>
<th>AC</th>
<th>BC</th>
</tr>
</thead>
<tbody>
<tr>
<td>CC</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

multiplicity \( n_i \): 2 1 2 4

energy \( h_i \): +J +J +J -J

probability \( p_i \): \( p_1 \) \( p_2 \) \( p_3 \) \( p_4 \)

\[ \sum n_i p_i = 2p_1 + p_2 + 2p_3 + 4p_4 = 1 \quad \text{(prob. normalized)} \]

\[ u = \sum n_i p_i h_i \quad \text{(energy constraint)} \]

single symbol probabilities:
\[ q_1 = P(A) = p_1 + p_3 + p_4 \]
\[ q_2 = P(B) = p_2 + 2p_4 \]
\[ q_3 = P(C) = p_1 + p_3 + p_4 \]

Check \( \sum q_j = \sum n_i p_i \); OK!

Now maximize \( s = S_2 - S_1 = \sum n_i p_i \ln \frac{1}{p_i} - \sum q_j h_i \ln \frac{1}{q_j} \)

subject to
\[ 1 = \sum n_i p_i \]

\[ u = \sum n_i p_i h_i \quad \text{(using Lagrangian optimizer)} \]
How long blocks do we need to consider?
2, because it is sufficient to require "only blank sites before and after married couples".

Configuration

\[
\begin{array}{cccc}
\text{Mult. } & n_i & 1 & 2 & 1 & 2 \\
\text{Prob. } & p_i & p_2 & p_3 & p_4 \\
\end{array}
\]

Single symbol probabilities:
\[
\begin{align*}
q_1 &= p(\square) = p_1 + p_2 + p_4 \\
n_2 &= p(\text{IT}) = p_2 + p_3 \\
n_3 &= p(\text{BA}) = p_4
\end{align*}
\]

Sum \[\sum q_j = p_1 + 2p_2 + p_3 + 2p_4 \quad \text{OK!}\]

Now maximize \[\Delta S_2 = \sum_{i=1}^{n} p_i n_i \ln \frac{1}{p_i} - \sum_{j=1}^{3} q_j \ln \frac{1}{q_j}\]
subject to
\[
1 = \sum_{i} p_i n_i
\]
\[
S = \frac{p(\text{IT}) + 2p(\text{BA})}{\sum p_i n_i}
\]
\[
\alpha = \frac{2p(\text{BA})}{p(\text{IT}) + 2p(\text{BA})}\]}