Simulation of Complex Systems

Homework 4: Forest Fires

Assessment date: 12th of December

Abstract

In this exercise we study a simple model of forests grown under exposure to forest fires, a model claimed to exhibit self-organized criticality. We analyze the size distribution of fires which seems to follow a power law.

Many models studied in statistical physics have critical points, where correlations diverge and things become complex and interesting in many ways. For the standard Ising model (in two or higher dimensions) this is the Curie point, T_C; as one approaches this point the typical size of domains (among other observables) diverges as a power law, $l \propto (T - T_C)^{-\alpha}$.

With this background it is natural that physicists find the observations of power laws in nature to be very exciting. We have seen some examples of degree distributions following power laws in the networks lectures, but many other quantities including word frequencies, severities of terrorist attacks, the populations of cities, individual wealth, and book sales have been claimed to do so as well.

While there are several mechanisms that give rise to power laws (including combinations of exponentials and random extremal processes), criticality is the most interesting one. However, most systems have to be carefully tuned to show criticality; in the Ising model we have to fix the temperature at T_C to see this behavior. This makes the critical scheme a less than convincing explanation for the common occurrence of power laws in nature.

A proposed solution is something called *self-organized criticality* (SOC). The idea is that something about the system drives it towards a critical point, regardless of the initial conditions. In the forest fire model [1] which we will study here, trees grow and fires burn down connected clusters of trees. Thus, when there are few trees, fires have little effect and the forest grows. However, when the density increases the fires tend to have more effect, burning down large swats of forest. There is thus a balancing act between the growing and burning effects, and the model is held by its own dynamics close to a critical point.

The model consist of a square $N \times N$–lattice on which the trees grow. Each site either has a tree or is empty. At each time step each empty site has a probability p to turn into an occupied one. Each time step also has a probability f of an lightning strike at a random site. The strike ignites the tree, if any, at that site. The tree ignites its von Neumann neighbors[1] (using periodic boundary conditions), they ignite their neighbors etc., and all ignited trees are removed from the lattice. In other words, the lightning strike starts a forest fire which burns down a connected cluster of trees (see Fig. 1). One can either have the fire to spread one step each time step or let the fire burn down the whole cluster during a single time step (corresponding to taking $p \to 0$ with p/f finite, separating the timescales for growth and burning completely). Use the latter.

The number of trees burnt down in a single strike is the size of the fire. We will study the distribution of these sizes.

Rank-frequency plots: Plotting power law distributions can be problematic as the tail gets very noisy and binning introduces artifacts. A solution is the rank-frequency plot. It effectively plots the complementary cumulative distribution[2] of the data, allowing easy comparison to the cCDF of the hypothetical distribution. To do such a plot given some data, we sort the data, giving a series $\{y_i\}_{i=1}^n$ with $y_i \geq y_{i+1}$, and assign to each value its (normalized) rank, giving a new data set $\{(y_i, i/n)\}_{i=1}^n$. This is the empirical cCDF, ready to be plotted in a log-log graph. See Fig. 1 for an example.

Generating power law data: There are several methods for generating random numbers with a given distribution $P(x)$. The most general is rejection sampling, but if one knows the CDF and can invert it, the more direct method of generating uniform numbers $r_i \sim \text{Unif}[0, 1]$ and picking $X_i = CDF^{-1}(r_i)$ is much more efficient, giving data with the desired distri-

[1]The nearest von Neumann neighbors of a cell is the four cells to its immediate right, left, top, and right.

[2]If a quantity has probability distribution $P(x)$, its complementary cumulative distribution is $cCDF(y) = \int_y^\infty P(x)dx$, that is, the probability of observing a value greater than or equal to y.

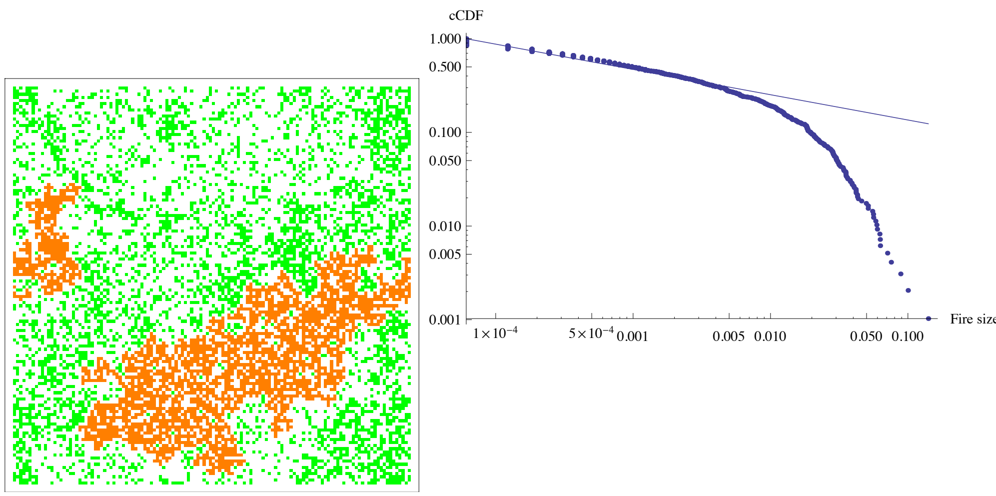

Figure 1: (Left) A snapshot of a forest where a cluster of 2605 trees is burning down. Green patches are occupied by trees, orange are burning. (Right) Rank-frequency plot of the fire sizes on a 128-by-128 lattice with a power law fitted to the bulk of the distribution.

bution $X_i \sim P(X_i)$. This approach is known as inverse transform sampling. For a power law distribution $P(x) \propto (x/x_{min})^{-\tau}$ the formula for the cCDF becomes $X_i = x_{min}(1 - r_i)^{-\frac{1}{\tau-1}}$.

Examination: Work in your assigned groups. During lab hour, either 12/12 or the extra lab 17/12, you should together demonstrate your results to a tutor in the way indicated at the respective exercise. (We might be a bit flexible here, in the sense that you can have your work assessed also in the other lab sessions if you have a compelling reason for doing so. But sticking to the schedule is the preferred options as that will make things run more smoothly). Also, when you have had your work assessed, email your code to `kolbjorn@chalmers.se` with "SoCS HP4" and your name in the subject.

Make sure you go through your demonstration by yourself before so that everything works. Everyone involved will appreciate the reduced queue times. Feel free to show just a subset of the exercises if you haven't done them all (whether you plan to do so later or not).

Exercises:

1. Implement the basic model and visualize it, see Fig. 1. What are reasonable values for the parameters? Note the shapes of the clusters removed by a lightning strike.
   To demonstrate: the visualization. **(6p)**

2. Study the size distribution of the fires on a fairly large lattice. For each fire, record its size and also compute the density of the forest just before the fire, generate a random forest (grown without fires) with a similar density and start a fire in it. Compare the rank-frequency plots of the distributions of fire sizes for the two types of forests.
   To demonstrate: the distributions (in a single plot). What is the interesting difference? **(7p)**

3. It is generally claimed that the self-organized fire distribution follows a power law with exponent $\tau = 1.15$ [2], $P(s) \propto s^{-1.15}$, in the limit of large systems. Try to evaluate this claim. Normally we should use maximum likelihood (MLE) to estimate the exponent [3] but in this case it works badly due to the cut-off of the distribution. An alternative would be to model the cut-off as an exponential and derive the MLE for the resulting distribution, but that is outside of the scope of

this exercise. Instead, to get a rough approximation, use the heretical method of drawing straight lines in your plot and calculate the slope (which is $1 - \tau$ in a rank-frequency plot). Check that this works acceptably by generating a synthetic data set (see above) from the power law and compare.
To demonstrate: the estimate for the exponent and a plot comparing the fire data and the synthetic power law data. **(7p)**

4. The value for the exponent depends on the size of the lattice N, so do a finite size analysis. Repeat the above exercise for a couple of N, plot the results vs. $1/N$, and extrapolate to zero (i.e. $N \to \infty$).
To demonstrate: the resulting plot with the extrapolation. **(5p)**

In most models studies such as the one we did above is enough (provided you did it on reasonably large lattices); increasing system size merely leads to more significant digits in the critical parameters. This model is different though. Grassberger [4] did extensive simulations up to $N = 65536$ (by representing each tree with a single bit, you can fit such a lattice into a 1 GB memory) and found new behavior for extremely large systems, with e.g. $\tau = 1.19 \pm 0.01$, noting that nothing prevents still new behavior at even larger scales. Looking at the density of trees, he also found that the previous result of $\rho \approx 0.408$ is an underestimate: the density continues to increase for ever larger systems, albeit very slowly. As the density cannot exceed the percolation density $\rho_c = 0.592...$, Grassberger conjectured that it will end up there. A simple extrapolation of his data gives that this point is reached somewhere around $N = 10^{20}\ldots$

References

[1] B. Drossel and F. Schwabl. Self-organized critical forest-fire model. *Physical Review Letters*, 69(11):1629–1632, 1992.

[2] P Grassberger. On a self-organized critical forest-fire model. *Journal of Physics A: Mathematical and General*, 26(9):2081–2089, May 1993.

[3] Aaron Clauset, Cosma Rohilla Shalizi, and M. E. J. Newman. Power-Law distributions in empirical data. *SIAM Review*, 51(4):661–703, January 2009.

[4] Peter Grassberger. Critical behaviour of the Drossel-Schwabl forest fire model. *New Journal of Physics*, 4(1):17, 2002.